Metrizability of spaces of holomorphic functions

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A B S T R A C T

In this paper we prove that if U is an open subset of a metrizable locally convex space E of infinite dimension, the space $\mathcal{H}(U)$ of all holomorphic functions on U, endowed with the Nachbin–Coeuré topology τ_δ, is not metrizable. Our result can be applied to get that, for all usual topologies, $\mathcal{H}(U)$ is metrizable if and only if E has finite dimension.

1. Introduction

The study of locally convex topologies on $\mathcal{H}(U)$ is a topic of interest for many researchers. It is very natural to ask which properties $\mathcal{H}(U)$ has from the point of view of functional analysis. In particular, some mathematicians have been interested in characterizing the locally convex spaces E such that $\mathcal{H}(U)$ is metrizable for all open subsets U of E and for topologies as the compact open, Nachbin ported and Nachbin–Coeuré topologies. In 1968, Alexander proved the following theorem for Banach spaces with Schauder basis, which was generalized by Chae:

Theorem 1. *(See [1, p. 13], [4, Theorem 16.10].) Let U be an open subset of an infinite dimensional Banach space. If τ is a topology on $\mathcal{H}(U)$ finer than the topology of pointwise convergence, then $(\mathcal{H}(U), \tau)$ is not metrizable.*

Although this result can be applied to all usual topologies on $\mathcal{H}(U)$, its proof is only valid when E is a Banach space because Baire and Josefson–Nissenzweig Theorems are used (see [6] and [11]).

In 2007, Ansemil and Ponte have got that, for the Nachbin ported topology, Theorem 1 can be generalized to metrizable locally convex spaces E (see [3]). In this article, we prove an analogous result for the Nachbin–Coeuré topology τ_δ, which answers a question stated by Mujica in [8, Problem 11.9] thirty years ago.

2. Definitions and previous results

Throughout this paper, the letter E will denote a complex locally convex space, E' will represent the dual space of E and U will be an open subset of E. A function $f : U \to \mathbb{C}$ is holomorphic on U if it is continuous and for each $z \in U$ and $w \in E$ the function of one complex variable

$$\lambda \mapsto f(z + \lambda w)$$

is holomorphic on a neighborhood of zero in $\mathbb{C}$. Let $\mathcal{H}(U)$ denote the space of all holomorphic functions on U. The compact open topology on $\mathcal{H}(U)$, τ_0, is defined by the seminorms

$$f \in \mathcal{H}(U) \mapsto \sup_{z\in K}\left|f(z)\right|$$

when K ranges over the compact subsets of U.

Let us recall the definition of other fundamental topologies on $\mathcal{H}(U)$. A seminorm p on $\mathcal{H}(U)$ is *ported* by a compact subset K of U if for every open neighborhood V of K in U there is a constant $C > 0$ such that

$$p(f) \leqslant C \cdot \sup_{z\in V}\left|f(z)\right|$$

for all $f \in \mathcal{H}(U)$. The Nachbin topology τ_ω is the locally convex topology on $\mathcal{H}(U)$ defined by the seminorms ported by the compact subsets of U.

The Nachbin–Coeuré topology, denoted by τ_δ, is the locally convex topology on $\mathcal{H}(U)$ defined by the seminorms p which verify the following property: for each increasing countable open cover of U, $\{V_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbb{N}$ and $C > 0$ such that

$$p(f) \leqslant C \cdot \sup_{z\in V_{n_0}}\left|f(z)\right|$$

for all $f \in \mathcal{H}(U)$. It is well known that the space $(\mathcal{H}(U), \tau_\delta)$ is bornological and $\tau_0 \leqslant \tau_\omega \leqslant \tau_\delta$ on $\mathcal{H}(U)$ (see [5, Propositions 3.17, 3.18]). Moreover, τ_ω and τ_δ coincide on E', which can be identified with a subspace of $\mathcal{H}(U)$ (see [5, Proposition 3.22]).

In our main result (Theorem 4), the following proposition will be used:

Proposition 2. *(See [3, Proposition 1].) Let E be a metrizable locally convex space. If (E', τ_ω) is metrizable, then E is a normed space.*

We will also need some results about bounding and limited sets. A subset A of E is said to be *bounding* if $\sup_{z\in A}|f(z)| < \infty$ for all $f \in \mathcal{H}(E)$. A set $B \subset E$ is *limited* if

$$\lim_{n\to\infty}\Bigl(\sup_{z\in B}\left|\varphi_n(z)\right|\Bigr) = 0$$

for every sequence $\{\varphi_n\}_{n=1}^{\infty} \subset E'$ such that $\lim_{n\to\infty}\varphi_n(z) = 0$ for all $z \in E$.

Proposition 3. *Let E be a Banach space.*

1. *Every bounding set in E is limited (see* [10, Corollary 2.13]*).*
2. *If A is a limited subset of E, the closed convex balanced hull of A is also limited (see* [10, Remark 4.2(c)]*).*
3. *If E has infinite dimension, every limited set in E has empty interior (see* [10, Corollary 4.13]*).*

The third property is a consequence of the Josefson–Nissenzweig Theorem. Indeed, if a limited subset A of an infinite dimensional Banach space E has no empty interior, there exist $z_0 \in A$ and $r > 0$ such that $B_E(z_0, r) \subset A$. Then $B_E(0,1) \subset \frac{1}{r}(A - z_0)$ and we obtain that $B_E(0,1)$ is also limited. By the Josefson–Nissenzweig Theorem, there is a sequence $\{\varphi_n\}_{n=1}^{\infty}$ in E' such that $\|\varphi_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\varphi_n(z) = 0$ for all $z \in E$. Therefore,

$$1 = \lim_{n\to\infty}\|\varphi_n\| = \lim_{n\to\infty}\Bigl(\sup_{z\in B_E(0,1)}\left|\varphi_n(z)\right|\Bigr) = 0.$$

This is absurd; so limited subsets in E have empty interior.

3. Metrizability of $\mathcal{H}(U)$

Theorem 4. *Let U be an open subset of a metrizable locally convex space E. If $(\mathcal{H}(U), \tau_\delta)$ is metrizable, then E is a finite dimensional normed space.*

Proof. The symbol $\widehat{E}$ will denote the completion of E. If $r > 0$, $B_E(0,r)$ and $B_{\widehat{E}}(0,r)$ will represent the open balls with center 0 and radius r in E and $\widehat{E}$, respectively.

If $(\mathcal{H}(U), \tau_\delta)$ is metrizable, then the subspace (E', τ_δ) is also metrizable. As the topologies τ_ω and τ_δ coincide on E', by Proposition 2 we obtain that E is a normed space. Its completion $\widehat{E}$ is a Banach space.

Let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be a fundamental system of neighborhoods of 0 in $(\mathcal{H}(U), \tau_\delta)$. For each $n \in \mathbb{N}$ let

$$A_n = \bigl\{z \in U\colon \left|f(z)\right| \leqslant 1 \text{ for every } f \in \mathcal{F}_n\bigr\}.$$

We claim that the sets A_n are limited in $\widehat{E}$. Indeed, if $n \in \mathbb{N}$ and $f \in \mathcal{H}(\widehat{E})$, then $f|_U$ is holomorphic on U. Since $\mathcal{F}_n$ is a neighborhood of zero, there is $\alpha > 0$ such that $\alpha f|_U \in \mathcal{F}_n$. If $z \in A_n$, then $|\alpha f(z)| \leqslant 1$ and so

$$\sup_{z \in A_n} \left| f(z) \right| \leqslant \frac{1}{\alpha} < \infty.$$

This shows that A_n is a bounding subset of $\widehat{E}$. Hence, by Proposition 3, A_n and $\overline{\Gamma(A_n)}^{\widehat{E}}$, the closed convex balanced hull of A_n, are limited in $\widehat{E}$ for every $n \in \mathbb{N}$.

If $z_0 \in U$, the mapping

$$T : \big(\mathcal{H}(U), \tau_\delta\big) \to \big(\mathcal{H}(U - z_0), \tau_\delta\big)$$

defined by

$$Tf(z) = f(z + z_0)$$

for each $f \in \mathcal{H}(U)$ and $z \in U - z_0$, is a homeomorphism. Therefore, the space $(\mathcal{H}(U), \tau_\delta)$ is metrizable if and only if $(\mathcal{H}(U - z_0), \tau_\delta)$ is also metrizable and so we can assume that $0 \in U$.

Now we use an adaptation of [7, p. 184] for open subsets made in [3, Theorem 4]. Let $r > 0$ such that $B_E(0, 2r) \subset U$. If $\hat{z} \in B_{\widehat{E}}(0, r)$, there is a point $z_1 \in B_E(0, r)$ such that

$$\|\hat{z} - z_1\| < \frac{r}{4}.$$

Therefore,

$$\hat{z} - z_1 \in B_{\widehat{E}}\left(0, \frac{r}{4}\right)$$

and there is $z_2 \in B_E(0, \frac{r}{4})$ such that

$$\|\hat{z} - z_1 - z_2\| < \frac{r}{4^2}.$$

If we repeat this argument, for each $n \in \mathbb{N}$ we can find a point $z_n \in B_E(0, \frac{r}{4^{n-1}})$ such that

$$\left\| \hat{z} - \sum_{k=1}^{n} z_k \right\| < \frac{r}{4^n}.$$

Hence $\hat{z} = \sum_{k=1}^{\infty} z_k$.

Let $w_n = 2^n z_n \in E$ for each $n \in \mathbb{N}$. The sequence $\{w_n\}_{n=1}^{\infty}$ converges to zero:

$$\|w_n\| = 2^n \|z_n\| < 2^n \cdot \frac{r}{4^{n-1}} = \frac{2r}{2^{n-1}} \xrightarrow[n \to \infty]{} 0.$$

Moreover,

$$w_n \in B_E\left(0, \frac{2r}{2^{n-1}}\right) \subset B_E(0, 2r) \subset U$$

for all n. Therefore,

$$K = \{w_n \colon n \in \mathbb{N}\} \cup \{0\}$$

is a compact subset of U and

$$\left\{ f \in \mathcal{H}(U) \colon \sup_{z \in K} \left| f(z) \right| \leqslant 1 \right\}$$

is a neighborhood of zero in $(\mathcal{H}(U), \tau_\delta)$. Since $\{\mathcal{F}_n\}_{n=1}^{\infty}$ is a fundamental system of neighborhoods of zero in $(\mathcal{H}(U), \tau_\delta)$, there is $n_1 \in \mathbb{N}$ such that

$$\mathcal{F}_{n_1} \subset \left\{ f \in \mathcal{H}(U) \colon \sup_{z \in K} \left| f(z) \right| \leqslant 1 \right\}.$$

Let $w \in K$. If $f \in \mathcal{F}_{n_1}$, then

$$\left| f(w) \right| \leqslant \sup_{z \in K} \left| f(z) \right| \leqslant 1.$$

Hence $w \in A_{n_1}$ and so $K \subset A_{n_1}$.

For every $n \in \mathbb{N}$, $\sum_{k=1}^{n} \frac{1}{2^k} w_k$ is a convex linear combination of elements of K, which implies that

$$\sum_{k=1}^{n} \frac{1}{2^k} w_k \in \Gamma(K) \subset \overline{\Gamma(A_{n_1})}^{\widehat{E}}.$$

As the last set is closed in $\widehat{E}$, we have

$$\hat{z} = \lim_{n\to\infty} \sum_{k=1}^{n} z_k = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} w_k \in \overline{\Gamma(A_{n_1})}^{\widehat{E}}.$$

Thus, we have proved that

$$B_{\widehat{E}}(0, r) \subset \bigcup_{n=1}^{\infty} \overline{\Gamma(A_n)}^{\widehat{E}}.$$

Since $B_{\widehat{E}}(0, r)$ is an open subset of the Banach space $\widehat{E}$, there exists $n_2 \in \mathbb{N}$ such that $\overline{\Gamma(A_{n_2})}^{\widehat{E}}$ has no empty interior in $\widehat{E}$. As we have seen that $\overline{\Gamma(A_{n_2})}^{\widehat{E}}$ is a limited set, by Proposition 3 $\widehat{E}$ has finite dimension and so E is also a finite dimensional space. □

Theorem 5. *Let U be an open subset of a metrizable locally convex space E. Let τ be a locally convex topology on $\mathcal{H}(U)$ such that $\tau_0 \leqslant \tau \leqslant \tau_\delta$. Then $(\mathcal{H}(U), \tau)$ is a metrizable space if and only if E has finite dimension.*

Proof. As E is a metrizable space, τ_δ is the bornological topology associated with τ_0 (see [5, Example 3.20(a)]). If E has finite dimension, it is well known that $(\mathcal{H}(U), \tau_0)$ is a metrizable space. Hence τ_0 is bornological and so $\tau_\delta = \tau_0$. This implies that $\tau = \tau_0$ and, therefore, $(\mathcal{H}(U), \tau)$ is metrizable.

Now we show the opposite implication. Since $\tau_0 \leqslant \tau \leqslant \tau_\delta$, τ_δ is also the bornological topology associated with τ. If $(\mathcal{H}(U), \tau)$ is metrizable, then τ is bornological and so $\tau = \tau_\delta$. Hence, by Theorem 4, E is a finite dimensional space. □

Theorem 5 can be applied to all usual topologies on $\mathcal{H}(U)$, among them, τ_ω, τ_b and β. The definition of τ_b is based on the topology of uniform convergence on bounded sets, while β is the strong topology when $\mathcal{H}(U)$ is considered as a dual space (see [5, Definitions 3.29 and 3.39]).

Our last proposition will show that the only hypothesis on Theorems 4 and 5 (E is metrizable) cannot be suppressed. We recall that a topological space X is said to be *hemicompact* if there is a fundamental sequence of compact subsets of X. A locally convex space E is a DFC space if there exists a Fréchet space F such that $E = (F', \tau_0)$.

Infinite dimensional DFC spaces are not metrizable. Indeed, if F is a Fréchet space and (F', τ_0) is metrizable, then there is a fundamental sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of F. Hence we have

$$F = \bigcup_{n=1}^{\infty} K_n.$$

Since F is a Baire space, there is $n \in \mathbb{N}$ such that K_n has no empty interior and, therefore, F is a finite dimensional space.

Proposition 6. *Let F be a separable Fréchet space and let U be an open subset of $E = (F', \tau_0)$. Then $\tau_0 = \tau_\delta$ on $\mathcal{H}(U)$ and $(\mathcal{H}(U), \tau_\delta)$ is a Fréchet space.*

Proof. Using the Banach–Dieudonné Theorem, it is possible to prove that (F', τ_0) is a k-space and so the open subset U is a k-space as well (see [9, Theorem 7.6]). Hence $(\mathcal{C}(U), \tau_0)$, the space of all continuous functions on U with the compact open topology, is complete and $(\mathcal{H}(U), \tau_0)$ is also complete because it is closed in $(\mathcal{C}(U), \tau_0)$.

Now we use [9, Theorem 7.4]. Let $\{V_m\}_{m=1}^{\infty}$ be a fundamental system of neighborhoods of 0 in F. Then the polar sets $\{V_m^\circ\}_{m=1}^{\infty}$ form a fundamental sequence of compact subsets of (F', τ_0). Since F is separable, there is a countable dense subset D in F and then the topology $\sigma(F', D)$ on F' is defined by a metric ρ. Moreover $\sigma(F', D)$ coincides with τ_0 on the compact subsets of U.

Consider the sets

$$L_{m,n} = \left\{ x \in V_m^\circ \cap U\colon\ \rho\big(x, V_m^\circ \backslash U\big) \geqslant \frac{1}{n} \right\},$$

where m and n are any natural numbers. Each $L_{m,n}$ is a compact subset of U because it is closed in V_m°. In the proof of [9, Theorem 7.4], Mujica asserts that $\{L_m = L_{m,m}\}_{m=1}^{\infty}$ is a fundamental system of compact subsets of U. As he has recognized in a private communication, it is not clear whether this is true. However, it is possible to prove that $\{L_{m,n}\colon\ m, n \in \mathbb{N}\}$ is a fundamental sequence of compact subsets of U and thus U is hemicompact. Hence the compact open topology on $\mathcal{H}(U)$ is metrizable and $(\mathcal{H}(U), \tau_0)$ is a Fréchet space.

As U is a k-space, τ_δ is the bornological topology associated with τ_0 (see [2, Theorem 1]). Since τ_0 is metrizable, it follows that τ_0 is bornological and then we have $\tau_0 = \tau_\delta$. $\square$

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